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Hyperbolic Systems with Relaxation: Characterization of Stiff Well-Posedness and Asymptotic Expansions

Jens Lorenz

E-mail: lorenz@math.unm.edu

and

H. Joachim Schroll

Numerical Mathematics, RWTH-Aachen, D-52056 Aachen, Germany and Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway
E-mail: schroll@math.ntnu.no.

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The Cauchy problem for linear constant-coefficient hyperbolic systems $u_t + \sum_j A^{(j)} u_{x_j} = (1/\delta)Bu + Cu$ in d space dimensions is analyzed. Here $(1/\delta)Bu$ is a large relaxation term, and we are mostly interested in the critical case where B has a non-trivial null-space. A concept of stiff well-posedness is introduced that ensures solution estimates independent of $0 < \delta \ll 1$. Stiff well-posedness is characterized algebraically and—under mild assumptions on B—is shown to be *equivalent* to the existence of a limit of the L_2-solution as $\delta \to 0$. The evolution of the limit is governed by a reduced hyperbolic system, the so-called equilibrium system, which is related to the original system by a phase speed condition. We also show that stiff well-posedness—which is a weaker requirement than the existence of an entropy—leads to the validity of an asymptotic expansion. As an application, we consider a linearized version of a generic model of two-phase flow in a porous medium and show stiff well-posedness using a general result on strictly hyperbolic systems. To confirm the theory, the leading terms of the asymptotic expansion are computed and compared with a numerical solution of the full problem.

Key Words: hyperbolic system; relaxation; well-posedness; asymptotic expansion; equilibrium limit; phase speed condition.

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1. INTRODUCTION

This paper is concerned with the Cauchy problem for multidimensional first-order systems with a large relaxation term,

$$\begin{aligned} &u_t + \sum_{j=1}^{d} A^{(j)} u_{x_j} = \frac{1}{\delta} Bu + Cu, \qquad 0 < \delta \ll 1,\, x \in \mathscr{R}^d,\, t \geq 0, \\ &u(x, t = 0, \delta) = f(x), \qquad x \in \mathscr{R}^d. \end{aligned} \tag{1}$$

Here $u(x, t, \delta) \in \mathscr{C}^n$, $f \in L_2(\mathscr{R}^d, \mathscr{C}^n)$ and $A^{(j)}, B, C$ are constant complex $n \times n$ matrices. The number d of space dimensions is not restricted. It is assumed that the system is *strongly hyperbolic*. This assumption is equivalent to well-posedness (in L_2) of the Cauchy problem (1) for every fixed positive δ. The concept of strong hyperbolicity, which is independent of the matrices B and C, is reviewed in the fist part of Section 2.

Even if (1) is strongly hyperbolic, the solution $u(x, t, \delta)$ may explode as $\delta \to 0$. It is precisely the concept of **stiff well-posedness**, as defined in Definition 2.1, which excludes such an explosion by postulating solution estimates independent of δ. Nevertheless, stiff well-posedness is not sufficient to ensure convergence of $u(x, t, \delta)$ as $\delta \to 0$ for general initial values in L_2. What is needed, in addition to stiff well-posedness, is the following **eigenvalue condition** for the matrix B

- All non-zero eigenvalues of B has negative real-parts, and if $\lambda = 0$ is an eigenvalue of B, then $\lambda = 0$ is semi-simple.

Then Theorem 2.2 states that both assumptions together, stiff well-posedness and the above eigenvalue condition for B, are necessary and sufficient for the existence of the L_2-limit of the solution $u(x, t, \delta)$ as $\delta \to 0$. Further discussions and implications of stiff well-posedness are given in Section 2. For example, an application of the Kreiss-matrix-theorem shows that stiff well-posedness is *equivalent* to the existence of a so-called symmetrizer, which is a bounded, positive definite Hermitian matrix function $H(\omega)$ suitably related to the symbol of the given differential operator. In general, the construction of such a symmetrizer $H(\omega)$ is non-trivial, however. Therefore, simple sufficient conditions for stiff well-posedness are called for. For strictly hyperbolic systems such conditions are given in Theorem 2.5.

In Section 3 we briefly discuss the relation of our approach, based on L_2-estimates, to the concept of entropies. The discussion is restricted to a simple setting in one space dimension. In this setting the existence of a quadratic entropy in the sense of Chen, Levermore, and Liu [1] is shown to be *equivalent* to the existence of a *constant* symmetrizer, $H(\omega) \equiv H_0$. A

2×2 example is presented which is stiffly well-posed, but which has no constant symmetrizer. Consequently, there is no quadratic entropy. The example shows that the requirement of stiff well-posedness is weaker than the requirement of the existence of an entropy.

The 2×2 example is somewhat degenerate. However, for systems of dimension 3 or larger (in any number of space dimensions) stiff well-posedness is a significantly weaker requirement than the existence of an entropy. We demonstrate this in Section 5, where we consider a linearized model for two-phase flow. The considerations lead to a whole class of 3×3 systems which are stiffly well-posed, but have no entropy. This is our main motivation to study the Cauchy problem (1) under the assumption of stiff well-posedness.

Whenever the above eigenvalue condition for B is satisfied, there is a transformation matrix T such that $\tilde{B} = T^{-1}BT = \operatorname{diag}(0, \tilde{B}_{22})$ is block diagonal and all eigenvalues of $\tilde{B}_{22}$ have strictly negative real parts. If one introduces new variables $v = v(x, t, \delta)$ by $Tv := u$, then the transformed system (1) takes the block form

$$v_t + \sum_{j=1}^{d} \begin{pmatrix} \tilde{A}_{11}^{(j)} & \tilde{A}_{12}^{(j)} \\ \tilde{A}_{21}^{(j)} & \tilde{A}_{22}^{(j)} \end{pmatrix} v_{x_j} = \frac{1}{\delta} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{B}_{22} \end{pmatrix} v + \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{pmatrix} v.$$

Here 0 represents zero matrices and $\tilde{A}^{(j)} = T^{-1}AT$ as well as $\tilde{C} = T^{-1}CT$ are partitioned as prescribed by the partitioning of $\tilde{B}$. We also partition $v = (v^I, v^{II})^T$ according to the block structure of $\tilde{B}$. Then, because the eigenvalues of $\tilde{B}_{22}$ have strictly negative real parts and $\delta > 0$ is small, it is plausible that v^{II} decays rapidly to zero with increasing t. Put differently, if $t > 0$ is fixed, one expects $v^{II}(x, t, \delta) \to 0$ as $\delta \to 0$. This convergence is in fact true under the assumption of stiff well-posedness. Moreover, $v^I(x, t, \delta)$ also converges, thus

$$\breve{v}(\cdot, t) = \lim_{\delta \to 0+} v(\cdot, t, \delta) = \begin{pmatrix} \breve{v}^I(\cdot, t) \\ 0 \end{pmatrix}, \qquad t > 0,$$

exists. Here $\breve{v}^I$ solves the so-called equilibrium system

$$\breve{v}_t^I + \sum_{j=1}^{d} \tilde{A}_{11}^{(j)} \breve{v}_{x_j}^I = \tilde{C}_{11} \breve{v}^I.$$

These results, as well as properties of the equilibrium system, are shown in Section 2.

In Section 4 we extend the convergence result and derive an asymptotic expansion

$$u = u_0 + \delta u_1 + \delta^2 u_2 + \cdots .$$

The leading term of the expansion, u_0, consists of the equilibrium limit and an initial layer. Here the layer (in time) is determined by a family of stiff ODEs; the space coordinate x is the family parameter. The higher order terms u_1, u_2, etc. have similar decompositions.

The formal process for obtaining the asymptotic expansion can be generalized to nonlinear problems with smooth solutions as demonstrated by Yong [9, 10]. Our contribution is to prove *validity* of the expansion under the assumption of still well-posedness. More precisely, we show that the error of the mth order approximation is $\mathscr{O}(\delta^{m+1})$ in maximum norm in any finite time interval,

$$\Delta u^{(m)} := u - \sum_{j=0}^{m} \delta^j u_j = \mathscr{O}(\delta^{m+1}), \qquad \delta \to 0.$$

In Section 5 the theory is applied to a linearized version of the following two-phase flow model arising in oil-recovery:

$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ (cs + a)_t + (cf(s,c))_x &= 0, \\ \delta a_t &= E(c) - a, \qquad 0 < \delta \ll 1. \end{aligned} \tag{2}$$

This system models the displacement of oil by a mixture of water and dissolved polymer in the sea ground, a so-called polymer flooding process. In this context s is the saturation of the aqueous phase. It is assumed that the void volume is filled with fluid, hence $1 - s$ is the saturation of the oleic phase. The concentration of the dissolved polymer in the aqueous phase is approximated by c, and a models the adsorption of the polymer into the sea ground. The adsorption reaction is fast compared to the characteristic speeds and, therefore, is represented by a stiff rate equation in the model. The equilibrium state of the adsorption process is given by $a = E(c)$, where E is a given smooth and increasing function. Finally, the fractional flow function $f(s,c)$ is a smooth, given function. Typically, when c is fixed, $f(\cdot, c)$ is an S-shaped function of s. This S-shape affects the hyperbolic character of the system and can cause problems for well-posedness of the initial value problem (cf. [3, 8]).

For the linearized system our theory applies. We determine sharp conditions on the parameters such that the system is stiffly well-posed though no quadratic entropy exists. To confirm validity of the asymptotic expansion, the leading terms are computed numerically and are compared with a numerical approximation of the full problem. In future work we plan to utilize the asymptotic expansion for numerical approximations.

Notations. With $\langle u, v\rangle$ and $|u|$ we denote the Euclidean inner product and norm in $\mathscr{C}^n$. The corresponding matrix norm is also denoted by $|\cdot|$. The L_2-norm of a vector function u: $\mathscr{R}^d \to \mathscr{C}^n$, for which every component is square integrable, is denoted by $\|u\|$. Similarly, if L: $L_2 \to L_2$ is a bounded linear operator, its norm is $\|L\|$. With $\|\cdot\|_{H_m}$ we denote the usual Sobolev-norm based on L_2,

$$\|u\|_{H_m}^2 = \sum_{|\beta|\le m} \|D^\beta u\|^2, \quad D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}}, \qquad |\beta| = \sum_{j=1}^{d} \beta_j,\ \beta \in \mathscr{N}_0^d.$$

If $H \in \mathscr{C}^{n\times n}$ is a positive definite Hermitian matrix, then

$$\langle u, v\rangle_H = \langle u, Hv\rangle, \qquad |u|_H^2 = \langle u, u\rangle_H$$

defines an inner product and a norm on $\mathscr{C}^n$. If Q_1 and Q_2 are Hermitian matrices, then we write $Q_1 \le Q_2$ if $\langle y, Q_1 y\rangle \le \langle y, Q_2 y\rangle$ for all $y \in \mathscr{C}^n$. By A^* we denote the complex conjugate transpose of a matrix A and

$$\operatorname{Re} A = \tfrac{1}{2}(A + A^*)$$

is the symmetric part of $A \in \mathscr{C}^{n\times n}$. With $\sigma(A)$ we denote the set of all eigenvalues of A.

2. THE CONCEPT OF STIFF WELL-POSEDNESS

Let us first recall the concept of well-posedness for non-stiff linear constant-coefficient systems. For every fixed positive δ the system (1) has the form

$$u_t + \sum_j A^{(j)} u_{x_j} = Cu. \tag{3}$$

Given an initial condition $u(x,0) = f(x)$, where

$$f \in M_0 = \left\{ f: \mathscr{R}^d \to \mathscr{C}^n \,\middle|\, f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathscr{R}^d} e^{i\langle x, k\rangle} \hat{f}(k)\, dk,\ \hat{f} \in C_0^\infty \right\},$$

the so-called M_0-solution of (3) is

$$u(x,t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathscr{R}^d} e^{i\langle k, x\rangle} e^{P(ik)t} \hat{f}(k)\, dk.$$

Here $P(ik) = C - i\sum_{j=1}^{d} k_j A^{(j)}$ is the symbol of the operator $C - \sum_j A^{(j)}\partial/\partial x_j$. The assignment $f \to u(\cdot, t)$ defines the M_0-solution operator $S_0(t)$: $M_0 \to M_0$, and we write $u(\cdot, t) = S_0(t)f$. Without making any assumption on the matrices $A^{(j)}$ and C in (3), the solution $u(x, t)$ depends analytically on (x, t), and $u(x, t)$ is a classical solution of (3). Conditions on $A^{(j)}$ enter the discussion, however, if one wants to extend $S_0(t)$ as a bounded linear operator from M_0 to L_2. We discuss this next, recalling the concept of strong hyperbolicity.

The system (3) is called *strongly hyperbolic* if for each $k \in \mathscr{R}^d$ the eigenvalues of $P_0(ik) = -i\sum_j k_j A^{(j)}$ are purely imaginary and $P_0(ik)$ can be diagonalized by a bounded transformation. Equivalently, there is a constant $c > 0$ and for each $k \in \mathscr{R}^d$ there is a matrix $H(k) \in \mathscr{C}^{n \times n}$ with

$$\frac{1}{c} I \le H(k) = H^*(k) \le cI, \tag{4}$$

and

$$\operatorname{Re}\big(H(k) P_0(ik)\big) = 0. \tag{5}$$

(See, for example, [4]). We refer to a matrix function $H(k)$ satisfying (4) and (5) as a symmetrizer for the operator $P_0 = -\sum_j A^{(j)}\partial/\partial x_j$.

Clearly, by definition, strong hyperbolicity is independent of the zero-order term Cu.

The system (3) is strongly hyperbolic if and only if the Cauchy problem for (3) is well-posed in L_2; i.e., there are constants K and α with

$$\|u(\cdot, t)\| \le K e^{\alpha t} \|f\|, \qquad t \ge 0, \tag{6}$$

or, equivalently,

$$\big|\exp\big[(C - P_0(ik))t\big]\big| \le K e^{\alpha t}, \qquad k \in \mathscr{R}^d,\ t \ge 0.$$

In (6), f denotes arbitrary initial data in M_0. Thus, under the assumption of strong hyperbolicity, the M_0-solution operator $S_0(t)$ is a bounded linear operator, $\|S_0(t)\| \le K e^{\alpha t}$, $t \ge 0$. As M_0 is dense in L_2, there is a unique L_2-solution of the Cauchy problem, obtained by extending $S_0(t)$ from M_0 to L_2. The L_2-solution $u(\cdot, t) = S(t)f$ satisfies the same bound (6) with general data $f \in L_2$. If (3) is strongly hyperbolic and $C = 0$, then one can choose $\alpha = 0$ in (6). In general, the value of α depends on C.

Now consider a stiff system,

$$u_t + A u_x = \frac{1}{\delta} B u, \qquad 0 < \delta \ll 1. \tag{7}$$

(For simplicity we first assume $C = 0$ and one space dimension.) Rescaling the variables $t = \delta\tau$ and $x = \delta\xi$, the explicit dependence on the parameter δ disappears,

$$u_\tau + Au_\xi = Bu.$$

Well-posedness of the Cauchy problem requires the existence of constants K and α with

$$\|u(\cdot, \tau)\| \le Ke^{\alpha\tau}\|u(\cdot, 0)\|, \qquad \tau \ge 0. \tag{8}$$

Rewriting (8) in terms of the original variables yields

$$\|u(\cdot, t, \delta)\| \le Ke^{\alpha t/\delta}\|u(\cdot, 0)\|, \qquad t \ge 0,\ \delta \in (0, 1].$$

Clearly, if t and α are strictly positive and δ is small, this bound becomes practically useless.

Therefore, the concept of stiff well-posedness requires a bound (8) without any exponential growth, i.e., with $\alpha = 0$. Equivalently,

$$|\exp[(B - i\omega A)\tau]| \le K, \qquad \omega \in \mathscr{R},\ \tau \ge 0.$$

Generalizing these considerations, we give the following definition.

DEFINITION 2.1. The Cauchy problem for a multidimensional system

$$u_t + \sum_{j=1}^{d} A^{(j)} u_{x_j} = \frac{1}{\delta} Bu + Cu, \tag{9}$$

where $A^{(j)}$, B, and C are constant matrices in $\mathscr{C}^{n\times n}$, is called stiffly well-posed, if there is a constant K such that

$$\left|\exp\left[\left(B - i\sum_{j=1}^{d} \omega_j A^{(j)}\right)\tau\right]\right| \le K, \qquad \forall\omega \in \mathscr{R}^d,\ \tau \ge 0. \tag{10}$$

Clearly, by definition, stiff well-posedness is independent of the term Cu in (9). Our reason for introducing the term Cu in (9) is that the treatment of such a term is very helpful if one wants to generalize our results to problems with variable coefficients. This extension will be given in future work. It is then important to know that the exponential growth rate, which the term Cu can generate, is independent of δ. This independence will be shown in Lemma 2.1 below.

Important necessary and sufficient conditions for the existence of a uniform bound (10) are given by the Kreiss-matrix-theorem. (See, for example, [4, Theorem 2.3.2].) The theorem implies, in particular, that stiff well-posedness is *equivalent* to the existence of a constant $c > 0$ and of a matrix function $H(\omega)$, defined for $\omega \in \mathscr{R}^d$ and taking values in $\mathscr{C}^{d \times d}$, such that

$$\frac{1}{c} I \le H(\omega) = H^*(\omega) \le cI \quad \text{and} \quad \mathrm{Re}\left(H(\omega)\left(B - i \sum_{j=1}^{d} \omega_j A^{(j)} \right)\right) \le 0. \tag{11}$$

Any such matrix function $H(\omega)$ will be called a symmetrizer for the stiff system (9).

From this formulation one easily derives a simple sufficient condition for stiff well-posedness. It is fulfilled if and only if one can choose a constant symmetrizer, $H(\omega) \equiv H_0$.

THEOREM 2.1. *Consider* (9) *and assume that there is a positive definite Hermitian matrix* H_0 *with* $H_0 A_j = A_j^* H_0$, $j = 1, \dots, d$, *and*

$$\mathrm{Re}(H_0 B) \le 0.$$

Then the Cauchy problem for this system is stiffly well-posed; *in particular*, *the system is strongly hyperbolic*.

In Section 3 we will relate this simple criterion to the existence of an entropy. If the Cauchy problem for (9) is stiffly well-posed, then the solutions have a limited exponential growth rate. (The rate of growth depends on C.) The converse is also true; i.e., if for some C the solutions of (9) have a growth rate independent of δ, then the Cauchy problem is stiffly well-posed. To show this, we use the notations

$$P_0(ik) = -i \sum_j k_j A^{(j)}, \qquad P(ik, \delta) = P_0(ik) + \frac{1}{\delta} B + C,$$

and recall that a solution estimate (where u solves (9)):

$$\|u(\cdot, t)\| \le K e^{\alpha t} \|u(\cdot, 0)\|, \qquad t \ge 0,\ \delta \in (0, 1], \tag{12}$$

with general initial data in L_2 is equivalent to a bound for the corresponding matrix exponentials,

$$|\exp(P(ik, \delta) t)| \le K e^{\alpha t}, \qquad k \in \mathscr{R}^d, t \ge 0,\ \delta \in (0, 1]. \tag{13}$$

LEMMA 2.1. *The Cauchy problem for system* (9) *is stiffly well-posed if and only if there are constants* K *and* α *such that* (13) *or* (12) *holds*.

Proof. First assume stiff well-posedness and let $H(\omega)$ satisfy (11). We set $H = H(\delta k)$ and obtain

$$\operatorname{Re} HP(ik, \delta) = \operatorname{Re} H\left(\frac{1}{\delta}B + P_0(ik) + C\right)$$
$$= \frac{1}{\delta}\operatorname{Re} H\big(B + P_0(i\delta k)\big) + \operatorname{Re} HC \leq \operatorname{Re} HC \leq \alpha H$$

for some α independent of k and δ. Such a matrix inequality, $HP + P^*H \leq 2\alpha H$, implies $H(P - \alpha I) + (P - \alpha I)^*H \leq 0$, which yields $|\exp[(P - \alpha I)t]_H \leq 1$. The bound (13) follows.

Now assume, conversely, that (13) holds for some K, α. An argument as above implies existence of new constants K', β with

$$\left|\exp\left[\left(\frac{1}{\delta}B + P_0(ik)\right)t\right]\right| \leq K'e^{\beta t}.$$

Introducing $t = \tau\delta$ and sending $\delta \to 0$ yields (10), and the lemma is proved.

So far, boundedness (or exponential growth at a rate independent of δ) of the solutions of (1) has been discussed. Clearly, this is not sufficient to imply convergence of the solutions $u(x, t, \delta)$ as $\delta \to 0$. As an example, consider the equation

$$u_t = \frac{i}{\delta}u, \qquad u(\cdot, 0, \delta) = f \in L_2.$$

The solution $u(x, t, \delta) = e^{it/\delta}f(x)$ is bounded, but does not converge as $\delta \to 0$. Obviously, one has to exclude purely imaginary non-zero eigenvalues of B. Interestingly, this simple condition on B together with stiff swell-posedness *characterizes* all strongly hyperbolic systems (9) for which the solution of the Cauchy problem converges as $\delta \to 0$, for all initial data $f \in L_2$. In the next theorem we give a precise formulation of this result.

THEOREM 2.2. *Assume that the system* (9) *is strongly hyperbolic*.

A. *Suppose that the following two conditions are satisfied.*

C1. *The Cauchy problem for system* (9) *is stiffly well-posed* (*see Definition* 2.1).

C2. *The matrix B has no purely imaginary eigenvalue different from zero*.

Then, for all $f \in L_2$ *and all* $t > 0$, *the limit of* $S(t, \delta)f$ *exists in* L_2 *as* $\delta \to 0$.

B. *Conversely, if for some* $t > 0$ *the limit of* $S(t, \delta)f$ *as* $\delta \to 0$ *exists in* L_2 *for every* $f \in L_2$, *then the above two conditions*, C1 *and* C2 *hold*.

For $C = 0$ and $d = 1$ this result is proven in [6]. Because the generalization is straightforward, we only sketch the proof of the above theorem below (following the statement of Theorems 2.3 and 2.4).

The next result shows how one can obtain the evolution of the limit $\breve{u}(\cdot, t) = \lim_{\delta \to 0} S(t, \delta) f$ if C1 and C2 are satisfied. In fact, we derive a so-called *equilibrium system*, which governs the evolution of the limit. First note that under assumption C1 of stiff well-posedness the condition C2 for B is equivalent to the following.

C3. All eigenvalues λ of B satisfy $\operatorname{Re} \lambda < 0$ or $\lambda = 0$. If $\lambda = 0$ is an eigenvalue of B, then $\lambda = 0$ is semisimple; i.e., the algebraic and geometric multiplicities of $\lambda = 0$ are the same.

(To see that C1 and C2 imply C3, consider (10) with $\omega = 0$.)

Under the sole assumption that C3 holds, we first *formally* derive the so-called equilibrium system. Afterward, assuming stiff well-posedness in addition to C3, we will show a convergence result. Furthermore, we will relate the phase speeds of the equilibrium system to those of the full system.

Assuming C3, there is a transformation $T \in \mathscr{C}^{n \times n}$ so that $T^{-1}BT$ has block form,

$$T^{-1}BT = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{B}_{22} \end{pmatrix} =: \tilde{B}. \tag{14}$$

Here $\tilde{B}_{22}$ is a square matrix whose eigenvalues are precisely the non-zero eigenvalues of B. In other words, all eigenvalues of $\tilde{B}_{22}$ have negative real part. Using T, we introduce new variables $v(x, t, \delta)$ and $g(x)$ by $u(x, t, \delta) = Tv(x, t, \delta)$ and $f(x) = Tg(x)$. Setting

$$\tilde{A}^{(j)} = T^{-1}A^{(j)}T = \begin{pmatrix} \tilde{A}_{11}^{(j)} & \tilde{A}_{12}^{(j)} \\ \tilde{A}_{21}^{(j)} & \tilde{A}_{22}^{(j)} \end{pmatrix}, \qquad \tilde{C} = T^{-1}CT = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{pmatrix},$$

one obtains the transformed system

$$v_t + \sum_{j=1}^{d} \tilde{A}^{(j)} v_{x_j} = \frac{1}{\delta} \tilde{B} v + \tilde{C} v, \qquad v(x, 0, \delta) = g(x).$$

We partition

$$v = \begin{pmatrix} v^I \\ v^{II} \end{pmatrix}$$

where the dimension of v^I is the dimension of the null-space of B. It is then plausible that the negativity of the spectrum of $\tilde{B}_{22}$ leads to rapid decay in time of v^{II} for small $\delta > 0$, and this suggests the limit system

$$\breve{v}_t^I + \sum_{j=1}^{d} \tilde{A}_{11}^{(j)} \breve{v}_{x_j}^I = \tilde{C}_{11} \breve{v}^I. \tag{15}$$

We call (15) the equilibrium system. Though (15) can be written down whenever C3 holds, one cannot expect convergence (for $\delta \to 0$) of $v^I = (T^{-1}u)^I$ to a solution of the equilibrium system unless one makes further assumptions. In fact, unless one assumes stiff well-posedness, one can generally not expect v^I to converge at all. In the following result we show that C1 and C2 (or, equivalently, C1 and C3) yield convergence.

THEOREM 2.3. *Consider the system* (9) *and assume that the conditions* C1 *and* C2 *are satisfied. Let $u = Tv$, $f = Tg$ denote the transformation described above with $f \in L_2$. Then, for any $t > 0$,*

$$v(\cdot, t, \delta) \to \begin{pmatrix} \breve{v}^I(\cdot, t) \\ 0 \end{pmatrix} \quad \text{as } \delta \to 0.$$

Here $\breve{v}^I(\cdot, t)$ is the L_2-solution of (15) *with initial data $\breve{v}^I(x, 0) = g^I(x)$, and convergence holds w.r.t. the L_2-norm. Furthermore, the system* (15) *is strongly hyperbolic, and for all $k \in \mathscr{R}^d$ all eigenvalues of $\sum_j k_j \tilde{A}_{11}^{(j)}$ lie between the minimal and maximal eigenvalue of $\sum_j k_j A^{(j)}$.*

Remark. For any wave vector k with $|k| = 1$ the eigenvalues of $\sum_j k_j A^{(j)}$ are the phase speeds of the full system (9). Plane waves travel at these speeds in direction k. Similarly, whenever (15) is strongly hyperbolic, the eigenvalues of $\sum_j k_j \tilde{A}_{11}^{(j)}$ for $|k| = 1$ are the phase speeds of the equilibrium system. Thus, by Theorem 2.3, assumptions C1 and C2 imply that the pair of systems (9) and (15) satisfies the following phase-speed condition: *For all direction vectors $k \in \mathscr{R}^d$, $|k| = 1$, the phase speeds* of (15) *lie between the corresponding minimal and maximal phase speeds of* (9). One can ask for a converse. More precisely, assume that C3 holds and that (9) and (15) are strongly hyperbolic. (This implies real phase speeds.) If the phase-speed condition holds, will C1 hold; i.e., will (9) be stiffly well-posed? The answer is no, in general. This has been shown by Example 4.4 in [6]. (The example is a system of three variables in one dimension).

We remark further that for constant coefficient strongly hyperbolic systems in one space dimension the phase speeds agree with the group velocities. The above phase speed condition is then nothing but the so-called subcharacteristic condition [1, 2, 5]. In multidimensions the group velocities generally differ from the phase speeds, however. Because of the convergence stated in Theorem 2.3, it is plausible that the group velocities of the equilibrium system are again included between corresponding group velocities of the full system. Precise statements will be given in future work.

The proofs of Theorems 2.2 and 2.3 are based on a corresponding convergence result for analytic solutions. Recall that M_0 denotes the space of all functions $f: \mathscr{R}^d \to \mathscr{C}^n$ whose Fourier transform is C^∞-smooth and compactly supported. Then, for arbitrary matrices $A^{(j)}$, $B \in \mathscr{C}^{n \times n}$; i.e., without requesting strong hyperbolicity, the initial value problem

$$\begin{aligned} u_t + \sum_{j=1}^{d} A^{(j)} u_{x_j} &= \frac{1}{\delta} Bu + Cu, \qquad 0 < \delta \ll 1,\ x \in \mathscr{R}^d,\ t \geq 0. \\ u(x, t = 0, \delta) &= f(x), \qquad x \in \mathscr{R}^d, \end{aligned} \tag{16}$$

with initial data $f \in M_0$ has an analytic solution, the so-called M_0-solution,

$$u(\cdot, t, \delta) = S_0(t, \delta) f \in M_0.$$

The following convergence result for M_0-solutions has been proven in [7].

THEOREM 2.4. *Assume* C3. *Then, for all $f \in M_0$ and all $t > 0$, the M_0-solution $u(\cdot, t, \delta) = S_0(t, \delta)f$ of* (16) *converges* (*w.r.t. the H_m-norm, for any m*) *to a limit $\breve{u}(\cdot, t)$ as $\delta \to 0$. It holds that $\breve{u}(\cdot, t) \in M_0$, and if $u = Tv$, $f = Tg$ denotes the transformation described above, then*

$$v(\cdot, t, \delta) \to \begin{pmatrix} \breve{v}^I(\cdot, t) \\ 0 \end{pmatrix} =: \breve{v}(\cdot, t) \quad \text{as } \delta \to 0 \quad \text{for } t > 0.$$

Here $\breve{u} = T\breve{v}$, and the function $\breve{v}^I$ is the M_0-solution of (15) *with initial data $\breve{v}^I(x, 0) = g^I(x)$.*

Under the assumption of Theorem 2.4 neither the full system (16) nor the equilibrium system (15) is necessarily hyperbolic, and even if both systems are strongly hyperbolic, the phase-speed condition might be violated.

Sketch of the Proof of Theorem 2.2, *Part A*. Due to the assumption of stiff well-posedness both solution operators, $S_0(t, \delta)$ and its L_2 extension $S(t, \delta)$ are bounded in norm by $Ke^{\alpha t}$. Let $f_j \in M_0$ denote a sequence approximating $f \in L_2$. For each M_0-solution $S_0(t, \delta)f_j$ the limit

$$\breve{u}_j = \lim_{\delta \to 0} S_0(t, \delta) f_j$$

exists by Theorem 2.4. Because of boundedness of $S_0(t, \delta)$ the sequence $\breve{u}_j$ is a Cauchy sequence in L_2, hence $\lim_{j \to \infty} \lim_{\delta \to 0} S_0(t, \delta) f_j$ exists in L_2. As $S(t, \delta)$ is bounded it follows that $S(t, \delta)f$ converges to $\breve{u}$ as $\delta \to 0$. In other words: The two limiting processes commute

$$\lim_{\delta\to 0}\lim_{j\to\infty} S(t,\delta)f_j = \lim_{j\to\infty}\lim_{\delta\to 0} S_0(t,\delta)f_j.$$

Concerning part B of Theorem 2.2 we have that for some $t^* > 0$ the limit $\lim_{\delta\to 0} S(t^*,\delta)f$ exists for all $f \in L_2$. The principle of uniform boundedness implies $\|S(t^*,\delta)\| \le K$ for $0 < \delta \le 1$. Therefore, the estimate of stiff well-posedness follows for large τ,

$$\left|\exp\left[\left(B - i\sum_{j=1}^{d}\omega_j A^{(j)}\right)\tau\right]\right| \le K, \qquad \forall\omega\in\mathscr{R}^d,\ \tau \ge \frac{t^*}{\delta} \ge t^*.$$

The bound on the remaining finite interval $0 \le \tau \le t^*$ follows from strong hyperbolicity. Necessity of C2 has been explained above already.

Sketch of the Proof of Theorem 2.3. Consider the initial data in the null-space of B:

$$f = Tg = T\begin{pmatrix} g^I(x) \\ o \end{pmatrix} \in M_0.$$

Stiff well-posedness implies that the solution is bounded in terms of the data

$$\|v(\cdot,t,\delta)\| \le Ke^{\alpha t}\|g^I\|.$$

By Theorem 2.4 we have $v(\cdot,t,\delta) \to (\breve{v}^I(\cdot,t),0)^T$ as $\delta\to 0$, and therefore $\|\breve{v}^I(\cdot,t)\| \le Ke^{\alpha t}\|g^I\|$. This estimate shows strong hyperbolicity of the equilibrium system.

The convergence claimed in Theorem 2.3 follows by approximating the data $f\in L_2$ by a sequence $f_j \in M_0$ and exchanging the limits $\delta\to 0$ and $j\to\infty$. It remains to prove the phase-speed condition, which follows directly from the corresponding result in one space dimension: Fix $k \in \mathscr{R}^d$ and consider the system

$$u_t + Au_x = \frac{1}{\delta}Bu, \qquad A = \sum_{j=1}^{d} k_j A^{(j)}.$$

For such a system in one dimension it is shown in [6] (by using the convergence already proved) that all eigenvalues of $\tilde{A}_{11} = \sum_j k_j \tilde{A}_{11}^{(j)}$ are bounded by the minimal and maximal eigenvalue of A.

We close this section by presenting a sufficient criterion for stiff well-posedness of *strictly* hyperbolic systems. This criterion will be used in

Section 5 to treat a linearized model for two-phase flow. Recall that a system

$$u_t + \sum_{j=1}^{d} A^{(j)} u_{x_j} = Cu$$

is called strictly hyperbolic if for all $\omega \in \mathscr{R}^d$, $\omega \neq 0$, the eigenvalues of $P_0(i\omega) = -i\sum_j \omega_j A^{(j)}$ are purely imaginary and distinct. This implies the existence of a bounded transformation, $|S(\omega)| + |S^{-1}(\omega)| \leq const.$, such that $S^{-1}P_0 S$ is diagonal. In particular, it follows that strictly hyperbolic systems are strongly hyperbolic.

We consider a stiff system in block form,

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \sum_{j=1}^{d} \begin{pmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{x_j} = \frac{1}{\delta} \begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{17}$$

with the corresponding equilibrium system

$$u_t + \sum_{j=1}^{d} A_{11}^{(j)} u_{x_j} = 0. \tag{18}$$

Let $Q(i\omega) := B - i\sum_j \omega_j A^{(j)} = \delta P(i\omega, \delta)$; then the following criterion holds.

THEOREM 2.5. *The Cauchy problem for* (17) *is stiffly well-posed if the following conditions are satisfied*:

(a) $\operatorname{Re} \lambda < 0 \ \forall \lambda \in \sigma(B_{22})$.

(b) *The system* $w_t + \sum_j A^{(j)} w_{x_j} = 0$ *is strictly hyperbolic*.

(c) *The equilibrium system* (18) *is strictly hyperbolic*.

(d) *For all* $\epsilon > 0$ *there is* $c_0 > 0$ *with the following property*: *If* $\epsilon \leq |\omega| \leq 1/\epsilon$ *and* $\lambda \in \sigma(Q(i\omega))$, *then either* $\operatorname{Re} \lambda = 0$ *and* λ *is semi-simple*, *or* $\operatorname{Re} \lambda \leq -c_0 < 0$.

Note that by Theorem 2.2 assumption (a) is necessary for the existence of the equilibrium limit. Assumption (d) requires somewhat more than the following condition, which is necessary for stiff well-posedness:

(d*) All eigenvalues of $Q(i\omega)$ satisfy $\operatorname{Re} \lambda \leq 0$, and if $\operatorname{Re} \lambda = 0$ then λ is semi-simple.

To discuss assumption (d) let $\lambda_j(\omega)$, $j = 1, \ldots, n$, denote the eigenvalues of $Q(i\omega)$; the eigenvalue functions $\lambda_j(\cdot)$ are continuous. Condition (d) is equivalent to the following: For $\omega \neq 0$ the eigenvalues separate into two groups,

$$\sigma_0(\omega) := \{\lambda_1(\omega), \ldots, \lambda_K(\omega) : \operatorname{Re} \lambda_j(\omega) = 0\},$$

and

$$\sigma_-(\omega) := \left\{ \lambda_{K+1}(\omega), \dots, \lambda_n(\omega) \colon \operatorname{Re} \lambda_j(\omega) < 0 \right\},$$

where the eigenvalues in $\sigma_0(\omega)$ are semi-simple and where K is independent of ω. Note that continuity of $\lambda_j(\cdot)$ for $j = K+1, \dots, n$ implies (d). (In examples, the two groups of eigenvalues, $\sigma_0(\omega)$ and $\sigma_-(\omega)$, will typically collide for $\omega \to 0$.)

Proof of Theorem 2.5. The goal is to bound the matrix exponential

$$\left| e^{Q(i\omega)\tau} \right| \le \text{const.} \tag{19}$$

uniformly for all $\omega \in \mathscr{R}^d$ and $\tau \ge 0$. We distinguish three different cases, namely, when $|\omega|$ is large, when $|\omega|$ is small, and the intermediate range.

(1) Let us consider the case of large ω first, $|\omega| \ge 1/\epsilon$. In this case $Q(i\omega)$ is dominated by $\sum_{j=1}^d \omega_j A^{(j)}$, and strict hyperbolicity of $w_t + \sum_j A^{(j)} w_{x_j} = 0$ can be exploited. We write

$$Q(i\omega) = |\omega| \left(\frac{1}{|\omega|} B - i \sum_{j=1}^{d} \omega'_j A^{(j)} \right), \qquad \omega' = \frac{\omega}{|\omega|}.$$

By assumption (b) the matrix $\sum_{j=1}^d \omega'_j A^{(j)}$ is diagonalizable,

$$S_0^{-1} \left(\sum_{j=1}^{d} \omega'_j A^{(j)} \right) S_0 = \Lambda_0, \qquad S_0 = S_0(\omega'),$$

and the real diagonal entries of Λ_0 have some positive distance. If $|\omega|$ is large enough, the eigenvalues of

$$\frac{1}{|\omega|} S_0^{-1} B S_0 - i \Lambda_0$$

are distinct as well and (by [6, Lemma 6.1], for example), there is a bounded transformation $S = S_0 + \mathscr{O}(1/|\omega|)$ such that $S^{-1} Q(i\omega) S$ is diagonal. By (d*)—which follows from (d)—the real part of this diagonal matrix is non-positive and hence the bound (19) follows for $|\omega| \ge 1/\epsilon$ and $\tau \ge 0$.

(2) Next consider small ω, $|\omega| \le \epsilon$. In this case the spectral condition (a) and strict hyperbolicity of the equilibrium system will be used. We write

$$Q(i\omega) = \begin{pmatrix} -i \sum \omega_j A_{11}^{(j)} & \mathscr{O}(|\omega|) \\ \mathscr{O}(|\omega|) & B_{22} + \mathscr{O}(|\omega|) \end{pmatrix}$$

and assume $\omega \neq 0$; for $\omega = 0$ the desired bound is clear. By assumption (a) the eigenvalues of both diagonal blocks are separated. Hence there is a bounded transformation $T = I + \mathscr{O}(|\omega|)$ such that

$$T^{-1}Q(i\omega)T = \begin{pmatrix} -i\sum \omega_j A_{11}^{(j)} + \mathscr{O}(|\omega|^2) & 0 \\ 0 & B_{22} + \mathscr{O}(|\omega|) \end{pmatrix}.$$

Furthermore, by (c) the matrix $\sum \omega_j' A_{11}^{(j)}$ with $\omega' = \omega/|\omega|$ can be diagonalized

$$S_1^{-1}\Big(\sum \omega_j' A_{11}^{(j)}\Big)S_1 = \Lambda_1,$$

and the diagonal entries of Λ_1 are distinct. Hence, if $|\omega|$ is small enough, there is another transformation $S = S_1 + \mathscr{O}(|\omega|)$, such that

$$S^{-1}\Big(-i\sum \omega_j' A_{11}^{(j)} + \mathscr{O}(|\omega|)\Big)S = \Lambda$$

is diagonal. Again, by (d*) we have $\operatorname{Re}\Lambda \leq 0$. Assumption (a) yields $\operatorname{Re}\sigma(B_{22} + \mathscr{O}(|\omega|)) \leq -c_1 < 0$, and the desired bound (19) follows for $|\omega| \leq \epsilon$ and $\tau \geq 0$.

(3) Now fix $\epsilon > 0$ small enough so that the arguments in (1) and (2) apply and consider $\epsilon \leq |\omega| \leq 1/\epsilon$. Clearly, in this case the essential assumption is (d). By Schur's theorem (cf. [4, Appendix 1]), there is a unitary transformation $U = U(\omega)$ so that

$$U^*Q(i\omega)U = \Lambda + R$$

is upper triangular. Here Λ is diagonal and R is strictly upper triangular. We may assume that the diagonal is ordered, $\Lambda = \operatorname{blockdiag}(\Lambda^I, \Lambda^{II})$, where $\operatorname{Re}\Lambda^I = 0$ and $\operatorname{Re}\Lambda^{II} \leq -c_0 I < 0$. Now we partition

$$R = \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix}$$

according to the block form of Λ. Note that by assumption (d) the eigenvalues of Q with $\operatorname{Re}\lambda = 0$ are semi-simple, thus $R_{11} = 0$. Furthermore, for bounded $|\omega|$ the matrix R is bounded. Because, by assumption, the spectra of Λ^I and $\Lambda^{II} + R_{22}$ are separated by $c_0 > 0$, Lemma 6.2 of [6] applies, and there is a bounded transformation $T = T(\omega)$ with

$$T^{-1}(\Lambda + R)T = \begin{pmatrix} \Lambda^I & 0 \\ 0 & \Lambda^{II} + R_{22} \end{pmatrix}.$$

Finally, because $\operatorname{Re}\Lambda^{II} \leq -c_0 I < 0$, the bound (19) follows from the boundedness of R_{22}. This completes the proof.

3. SYMMETRIZERS VERSUS ENTROPIES

Entropies and symmetrizers are both well-established tools for the study of hyperbolic PDEs and also for systems of other types. In the case of a constant coefficient system, a symmetrizer is often constructed in Fourier space. Using the tools of pseudo-differential operators, the construction can then be extended to variable coefficient problems with the aim to obtain a norm in which the solution can be controlled.

In contrast, entropies are typically constructed directly in terms of physical variables. The aim is, again, to obtain a functional of the solution which can be controlled. In this section we relate symmetrizers and entropies in the simplest setting.

For a system in one space dimension,

$$u_t + Au_x = 0,$$

a symmetrizer is, by definition, a matrix function $H = H(k)$, $k \in \mathscr{R}$, with the following properties:

(1) There exists $c > 0$ with $c^{-1}I \le H(k) = H^*(k) \le cI$, $k \in \mathscr{R}$.

(2) $\mathrm{Re}(H(k)P(ik)) = 0$, $k \in \mathscr{R}$, where $P(ik) = -ikA$.

For $k \neq 0$, the condition $\mathrm{Re}(H(k)P(ik)) = 0$ is equivalent to

$$(H(k)A)^* = H(k)A. \tag{20}$$

Therefore, if no additional properties of $H(k)$ are needed, it is natural to assume that $H(k) = H_0$ does not depend on k. Let us assume this and let us assume, for simplicity, that A and H_0 are real. Then (20) requires *symmetry* of the product $H_0 A$.

Recall that a smooth scalar function φ: $\mathscr{R}^n \to \mathscr{R}$ is called an *entropy* for the system of conversation laws

$$u_t + f(u)_x = 0, \qquad x \in \mathscr{R}, \tag{21}$$

if the matrix $\varphi''(u)f'(u)$ is symmetric for all $u \in \mathscr{R}^n$. (Here f: $\mathscr{R}^n \to \mathscr{R}^n$ is a flux function with Jacobian $f'(u)$ and $\varphi''(u)$ is the Hessian of φ.) To explain the significance of the symmetry of $\varphi''(u)f'(u)$, let φ: $\mathscr{R}^n \to \mathscr{R}$ denote an arbitrary smooth function and consider the vector field $V(u) = \varphi'(u)f'(u)$ with Jacobian $V'(u) = \varphi''(u)f'(u) + \varphi'(u)f''(u)$. The term $\varphi'(u)f''(u)$ is always symmetric and, consequently, the symmetry of $\varphi''(u)f'(u)$ is *equivalent* to the symmetry of $V'(u)$. Therefore, if φ: $\mathscr{R}^n \to \mathscr{R}$ is an entropy, then $V'(u)$ is symmetric, and the vector field $V(u)$ has a potential $\psi(u)$,

$$\varphi'(u)f'(u) = V(u) = \psi'(u).$$

It is easy to show that the function $\psi(u)$, called an entropy flux, has the following property: If $u(x,t)$ is a smooth solution of (21), then $\varphi(u)_t + \psi(u)_x = 0$. Under appropriate assumptions, this fundamental relation is then used to obtain estimates for $\varphi(u(x,t))$ and then for u.

Now consider the simple case where $f(u) = Au$, $u \in \mathscr{R}^n$, is linear and

$$\varphi(u) = \tfrac{1}{2} u^T H_0 u, \qquad u \in \mathscr{R}^n, \tag{22}$$

is quadratic. (We assume here that A and $H_0 = H_0^T$ are real.) Then $\varphi''(u) f'(u) = H_0 A$; i.e., $\varphi(u)$ is an entropy if and only if $H_0 A$ is symmetric. Clearly, if $H_0 = H_0^T$ is a positive definite matrix (as is required for a symmetrizer), then $\varphi(u)$ is a positive definite quadratic form, and in particular $\varphi(u)$ is strictly convex. If (22) is an entropy, then

$$\psi(u) = \tfrac{1}{2} u^T H_0 A u$$

is the corresponding entropy flux.

Now consider

$$u_t + A u_x = \frac{1}{\delta} B u, \qquad 0 < \delta \le 1. \tag{23}$$

If we assume, as above,

$$H_0 = H_0^T, \qquad H_0 A = (H_0 A)^T,$$
$$\varphi(u) = \tfrac{1}{2} u^T H_0 u,$$
$$\psi(u) = \tfrac{1}{2} u^T H_0 A u,$$

then we obtain

$$\varphi(u)_t + \psi(u)_x = u^T H_0 u_t + u^T H_0 A u_x = \frac{1}{\delta} u^T H_0 B u$$

for any smooth solution $u(x,t)$ of (23). If we make the requirement

$$H_0 B + B^* H_0 \le 0, \tag{24}$$

which yields stiff well-posedness by Theorem 2.1, we obtain the entropy inequality $\varphi(u)_t + \psi(u)_x \le 0$ for all smooth solutions of (23). Thus (24) is consistent with an entropy inequality.

Our discussion has shown that Statement 1 below implies Statements 2 and 3:

STATEMENT 1. *There exists a symmetric, positive definite matrix* H_0 *so that* $H_0 A$ *is symmetric and* $H_0 B + B^T H_0 \le 0$.

STATEMENT 2. *The Cauchy problem for the family of strongly hyperbolic systems* (23) *is stiffly well-posed.*

STATEMENT 3. *There exists a strictly convex, quadratic function* $\varphi\colon \mathscr{R}^n \to \mathscr{R}$ *such that* $\varphi''(u)A$ *is symmetric* (*thus* φ *is an entropy*) *and a smooth function* $\psi\colon \mathscr{R}^n \to \mathscr{R}$ (*an entropy flux*) *with* $\psi''(u) = \varphi'(u)A$ *and* $\varphi(u)_t + \psi(u)_x \le 0$ *for all smooth solutions of* (23).

Clearly Statement 3 implies Statement 1, because a quadratic, strictly convex entropy is given by $\varphi(u) = \frac{1}{2}u^T H u$ where H is a symmetrizer for A and

$$\varphi(u)_t + \psi(u)_x = u^T H u_t + u^T H A u_x = \frac{1}{\delta} u^T H B u$$

$$= \frac{1}{2\delta} u^T (HB + B^T H) u \le 0.$$

We want to point out that the existence of a strictly convex, quadratic entropy (Statement 3) is equivalent to our simple criterion for stiff well-posedness (Statement 1). In general, however, stiff well-posedness is a weaker assumption than the requirements in Statements 1 or 3. In other words: Statements 1 and 3 are equivalent sufficient conditions for stiff well-posedness, but they are not necessary.

The following 2×2 system is an example of a system for which the Cauchy problem is stiffly well-posed, but no constant symmetrizer H_0 with $\mathrm{Re}(H_0 B) \le 0$ exists and hence no quadratic, strictly convex entropy exists:

$$u_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u_x = \frac{1}{\delta}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} u. \tag{25}$$

A constant symmetrizer H for $\Lambda = \mathrm{diag}(1, -1)$ has to be diagonal. This follows from $H\Lambda = \Lambda H$ and $H = H^*$. Without loss of generality we may assume

$$H = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}, \qquad h > 0.$$

For

$$B = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

it follows that

$$HB = \begin{pmatrix} 0 & 1 \\ 0 & -h \end{pmatrix}$$

and

$$HB + B^T H = \begin{pmatrix} 0 & 1 \\ 1 & -2h \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_{\pm} = -h \pm \sqrt{h^2 + 1} \in \mathscr{R}$. Because they have opposite signs the matrix is indefinite. This shows that the equivalent Statements 1 and 3 do not hold for the system (25).

Nevertheless, the Cauchy problem for this system is stiffly well-posed as follows from Theorem 1.3 in [6]. Because the system (25) is strictly hyperbolic, we can also apply the sufficient criterion Theorem 2.5: In block-form, system (25) reads

$$v_t + \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} v_x = \frac{1}{\delta} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} v.$$

The equilibrium equation is the scalar advection equation $\breve{v}_t^I + \breve{v}_x^I = 0$. The eigenvalues $\lambda_{1,2}$ of

$$Q(i\omega) = \begin{pmatrix} -i\omega & -2i\omega \\ 0 & -1 + i\omega \end{pmatrix}$$

are obviously $\lambda_1 = -i\omega$ and $\lambda_2 = -1 + i\omega$. Because $\operatorname{Re} \lambda_1 \equiv 0$ and $\operatorname{Re} \lambda_2 \equiv -1$ for all $\omega \in \mathscr{R}$, the Cauchy problem for (25) is stiffly well-posed.

4. ASYMPTOTIC EXPANSION

In this section we derive an asymptotic expansion of the solution of (1), assuming only that the system is stiffly well-posed and that C2 holds. For simplicity of presentation we will omit the bounded source term Cu and will restrict ourselves to the case of one space variable. Without loss of generality we also assume that the matrix B is already in block form. Thus we consider a system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = \frac{1}{\delta} \begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F \\ G \end{pmatrix}, \tag{26}$$

with initial condition

$$\begin{pmatrix} u \\ v \end{pmatrix}(x, 0, \delta) = \begin{pmatrix} f \\ g \end{pmatrix}(x), \qquad x \in \mathscr{R},\ \delta \in (0, 1]. \tag{27}$$

Here

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix}$$

are constant complex $n \times n$ matrices. We introduce forcing terms, F and G, into the system because even if we assume no forcings, inhomogeneous equations like (26) appear in the process of the asymptotic expansion. Because this section is essentially self-contained, the change of notation should cause no problems.

The assumption of stiff well-posedness requires:

(a) There is a constant $c_1 > 0$ such that $|\exp(B - i\omega A)\tau)| \leq c_1$ for all $\omega \in \mathscr{R}$ and $\tau \geq 0$.

Furthermore, stiff well-posedness and C2 yield:

(b) $\operatorname{Re} \lambda \leq -\gamma < 0$ for all $\lambda \in \sigma(B_{22})$.

Let $T > 0$ denote some fixed time. For simplicity of presentation we make strong smoothness assumptions for the data $h(x) = (f, g)^T(x)$, $H(x,t) = (F, G)^T(x,t)$ and require that

$$h \in C_0^\infty(\mathscr{R}); \qquad H \in C^\infty(\mathscr{R} \times [0, T]).$$

Also, assume that there is a $\kappa \geq 0$ with

$$H(x,t) = 0 \quad \text{if } |x| \geq \kappa, 0 \leq t \leq T.$$

Thus, the data are C^∞-smooth and have a compact support in x, which is uniform for $0 \leq t \leq T$. Then, because of the finite speed of propagation in hyperbolic systems, which is independent of δ, the solution and all approximating functions appearing below are also C^∞-smooth and compactly supported in some common x-region. (One can relax the smoothness requirement for the data and the assumption of a compact support by counting derivatives and requiring sufficiently strong decay as $|x| \to \infty$.)

For sufficiently regular vector functions φ: $\mathscr{R} \times [0, T] \to \mathscr{R}^k$ we use the notations

$$M_j(\varphi) = \max_{0 \leq t \leq T} \| \partial_x^j \varphi(\cdot, t) \|, \quad I_j(\varphi) = \int_0^T \| \partial_x^j \varphi(\cdot, t) \| \, dt, \qquad j = 0, 1, \ldots .$$

Here $\partial_x^j = \partial^j / \partial x^j$. (Thus, M_j takes a maximum over $0 \leq t \leq T$, whereas I_j takes an integral. We always use L_2-norms over space.) Setting $w = (u, v)^T$ the initial value problem (26), (27) reads

$$w_t + A w_x = \frac{1}{\delta} B w + H(x,t), \qquad w(x, 0, \delta) = h(x). \tag{28}$$

As above, $S(t, \delta)$ denotes the solution operator of $w_t + Aw_x = \frac{1}{\delta}Bw$. Then, by Duhamel's principle it follows that

$$w(\cdot, t, \delta) = S(t, \delta)h + \int_0^t S(t - \xi, \delta)H(\cdot, \xi)\, d\xi.$$

Assumption (a) yields $\|S(t, \delta)\| \leq c_1$ for all $t \geq 0$, and therefore

$$\|w(\cdot, t, \delta)\| \leq c_1\|h\| + c_1\int_0^t \|H(\cdot, \xi)\|\, d\xi,$$

thus

$$\max_{0 \leq t \leq T} \|w(\cdot, t, \delta)\| \leq c_1\{\|h\| + I_0(H)\}.$$

If we differentiate (28) j-times with respect to x, we obtain

$$\max_{0 \leq t \leq T} \left\| \partial_x^j w(\cdot, t, \delta) \right\| \leq c_1\left\{\|\partial_x^j h\| + I_j(H)\right\}.$$

LEMMA 4.1. *Under the smoothness assumption stated above, the solution of the forced stiff system* (28) *is bounded in terms of the data as*

$$M_j(w(\cdot, \cdot\,, \delta)) \leq c_1\left\{\|\partial_x^j h\| + I_j(H)\right\}, \qquad \delta \in (0, 1],\ j \in \mathcal{N}_0.$$

After these preliminary considerations, we derive an asymptotic expansion.

4.1. *Recursive Definition of the Expansion Terms*

Because of assumption (b) we expect v to decay like $e^{-\gamma\tau}$, where $\tau = t/\delta$ is the fast time variable. Therefore the following ansatz is reasonable

$$w(x, t, \delta) = \sum_{l=0}^{m} \delta^l w_l(x, t) + \sum_{l=0}^{m} \delta^l W_l\left(x, \frac{t}{\delta}\right) + \Delta w^{(m)}(x, t, \delta). \quad (29)$$

Here $\sum_l \delta^l w_l$ is the outer approximation; the inner approximation $\sum_l \delta^l W_l$ describes an initial layer. The ansatz is standard. Our point is to prove that under minimal structural assumptions—stiff well-posedness and C2—the error $\Delta w^{(m)}$ is of order δ^{m+1} in maximum norm.

Substituting (29) into the Cauchy problem (26), (27) and collecting terms multiplied by equal powers of δ, one obtains defining equations for the expansion terms $w_l = (u_l, v_l)^T$ and $W_l = (U_l, V_l)^T$.

The leading order terms are determined as

$$U_0 = 0, \tag{a-0}$$

$$v_0 = 0, \tag{b-0}$$

$$\partial_t u_0 + A_{11}, \partial_x u_0 = F, \qquad u_0(x,0) = f(x), \tag{c-0}$$

$$\partial_\tau V_0 = B_{22} V_0, \qquad V_0(x,0) = g(x). \tag{d-0}$$

Thus, u_0 is given by the equilibrium system, which is strongly hyperbolic by Theorem 2.3. From Lemma 4.1 it follows that

$$M_j(u_0) \le K\left\{\|\partial_x^j f\| + I_j(F)\right\} \le C.$$

Here and throughout this section C denotes a generic constant, which may depend on the data f, g, F, G and its derivatives, but not on δ. Using the PDE satisfied by u_0, time derivatives can be expressed by space derivatives. Therefore,

$$M_j\left(\partial_t^k u_0\right) \le M_{j+k}(u_0) + M_j\left(\partial_t^k F\right) \le C.$$

Because of (b), the initial layer $V_0(x,\tau) = e^{B_{22}\tau} g(x)$ is bounded by

$$\left\| \partial_x^j V_0(\cdot,\tau) \right\| \le C \|\partial_x^j g\| e^{-\gamma\tau} \le C e^{-\gamma\tau}.$$

Using the ODE $\partial_\tau V_0 = B_{22} V_0$ it follows that

$$\left\| \partial_x^j \, \partial_\tau^k V_0(\cdot,\tau) \right\| \le C e^{-\gamma\tau}.$$

Having determined w_{m-1} and W_{m-1}, the next terms are determined by

$$\partial_\tau U_m = -A_{11}\,\partial_x U_{m-1} - A_{12}\,\partial_x V_{m-1}, \qquad U_m(x,\infty) = 0, \tag{a-m}$$

$$B_{22} v_m = A_{21}\,\partial_x u_{m-1} + A_{22}\,\partial_x v_{m-1} + \partial_t v_{m-1} - \begin{cases} G, & m = 1, \\ 0, & m > 1, \end{cases} \tag{b-m}$$

$$\partial_t u_m + A_{11}\,\partial_x u_m = -A_{12}\,\partial_x v_m, \qquad u_m(x,0) = -U_m(x,0), \tag{c-m}$$

$$\partial_\tau V_m = B_{22} V_m - A_{21}\,\partial_x U_{m-1} - A_{22}\,\partial_x V_{m-1}, \qquad V_m(x,0) = -v_m(x,0). \tag{d-m}$$

Let us summarize the constructions, which naturally extend to any number of space dimension.

- $U_m(x,\tau)$ is determined by integration from $\tau = \infty$ to $\tau = 0$. This function "generates" initial data $U_m(x,0)$, which are eliminated by u_m.

- $v_m(x,t)$ is determined by inverting B_{22}. This generates initial data $v_m(x,0)$, which are eliminated by V_m.
- $u_m(x,t)$ is determined by an inhomogeneous equilibrium system.
- $V_m(x,\tau)$ is determined by an ODE initial value problem.

The following bounds will be used to show the error estimate below.

LEMMA 4.2. *Assume the data are smooth* (*see above*), *the Cauchy problem for system* (26) *is stiffly well-posed and* $\operatorname{Re}\sigma(B_{22}) \le -\gamma < 0$. *Let* $(u_m, v_m)^T$ *and* $(U_m, V_m)^T$ *be recursively defined by* (a-m)–(d-m). *Then the following holds*

(i) $\|\partial_x^j\, \partial_\tau^k U_m(\cdot,\tau)\| \le e^{-\gamma\tau} C \sum_{l=0}^{m-1} \tau^l$.

(ii) $M_j(\partial_t^k v_m) \le C$.

(iii) $M_j(\partial_t^k u_m) \le C$.

(iv) $\|\partial_x^j\, \partial_\tau^k V_m(\cdot,\tau)\| \le e^{-\gamma\tau} C \sum_{l=0}^{m} \tau^l$.

Here the constant C depends on the data and their derivatives up to some finite order, determined by m.

Proof. Assume the statements hold for some $m \ge 0$. We prove it for $m+1$.

Integrating backward in time, we find that

$$U_{m+1}(x,\tau) = \int_\tau^\infty A_{11}\, \partial_x U_m(x,\xi) + A_{12}\, \partial_x V_m(x,\xi)\, d\xi,$$

and hence

$$\partial_x^j U_{m+1}(x,\tau) = \int_\tau^\infty A_{11}\, \partial_x^{j+1} U_m(x,\xi) + A_{12}\, \partial_x^{j+1} V_m(x,\xi)\, d\xi.$$

Using the induction assumption it follows that

$$\left\| \partial_x^j U_{m+1}(\cdot,\tau) \right\| \le C \int_\tau^\infty e^{-\gamma\xi} \sum_{l=0}^{m} \xi^l\, d\xi.$$

Integration by parts yields

$$\left\| \partial_x^j U_{m+1}(\cdot,\tau) \right\| \le e^{-\gamma\tau} C \sum_{l=0}^{m} \tau^l.$$

This is statement (i) for $k=0$. If $k \ge 1$, then the same bound follows directly from the ODE and the induction assumption.

Next, we consider v_{m+1}, which is explicitly given by u_m, v_m and, if $m = 1$, by G. Therefore, the bound (ii) follows directly from the assumption.

The term u_{m+1} solves the equilibrium system with a modified forcing. Hence Lemma 4.1 applies and yields that

$$M_j(u_{m+1}) \le C.$$

Again, using the PDE $\partial_t u_m + A_{11}\,\partial_x u_m = -A_{12}\,\partial_x v_m$, time derivatives can be expressed by space derivatives, and (iii) follows for u_{m+1}.

It remains to bound V_{m+1}. Using Duhamel's principle it holds that

$$\begin{aligned} V_{m+1}(x,\tau) &= -e^{B_{22}\tau} v_m(x,0) \\ &\qquad - \int_0^\tau e^{B_{22}(\tau-\xi)}\big(A_{21}\,\partial_x U_m(x,\xi) + A_{22}\,\partial_x V_m(x,\xi)\big)\,d\xi. \end{aligned}$$

By assumption (b) it follows that

$$\|V_{m+1}(\cdot,\tau)\| \le e^{-\gamma\tau}C + e^{-\gamma\tau}C\int_0^\tau \sum_{l=0}^{m} \xi^l\,d\xi \le e^{-\gamma\tau}C\sum_{l=0}^{m+1}\tau^l.$$

The corresponding bound for mixed space and time derivatives follows in the same way by applying the derivatives to V_{m+1}. This concludes the proof of the lemma.

4.2. *The Error-Estimate*

Because we are dealing with linear problems, the equations for the error $\Delta w^{(m)} = w - w^{(m)}$ are obtained by subtracting the equation for $w^{(m)}$ from the original system (26). For the sum $u^{(m)} = \sum_{k=0}^{m} \delta^l(u_l + U_l)$ it holds that

$$\begin{aligned} u_t^{(m)} + A_{11}u_x^{(m)} + A_{12}v_x^{(m)} &= F(x,t) - \delta^{m+1}\,\partial_t U_{m+1}\left(x, \frac{t}{\delta}\right), \\ u^{(m)}(x,0,\delta) &= f(x). \end{aligned}$$

Furthermore, $v^{(m)}$ satisfies

$$\begin{aligned} v_t^{(m)} + A_{21}u_x^{(m)} + A_{22}v_x^{(m)} &= \frac{1}{\delta}B_{22}v^{(m)} + G(x,t) + \delta^m B_{22}(v_{m+1} + V_{m+1}) \\ &\qquad - \delta^{m+1}\,\partial_t V_{m+1}\left(x, \frac{t}{\delta}\right), \\ v^{(m)}(x,0,\delta) &= g(x). \end{aligned}$$

As the systems are linear, the equations for the errors $\Delta u^{(m)} = u - u^{(m)}$, $\Delta v^{(m)} = v - v^{(m)}$ are

$$\begin{aligned} \Delta u_t^{(m)} + A_{11}\Delta u_x^{(m)} + A_{12}\Delta v_x^{(m)} &= \delta^{m+1}\,\partial_t U_{m+1}, \\ \Delta v_t^{(m)} + A_{21}\Delta u_x^{(m)} + A_{22}\Delta v_x^{(m)} &= \frac{1}{\delta}B_{22}\Delta v^{(m)} + \delta^{m+1}\,\partial_t V_{m+1} \\ &\quad - \delta^m B_{22}(v_{m+1} + V_{m+1}), \\ \Delta u^{(m)}(x,0,\delta) = \Delta v^{(m)}(x,0,\delta) &= 0. \end{aligned}$$

At this stage an application of Lemma 4.1 would yield an error estimate of order δ^m, because of the forcing $\delta^m B_{22}(v_{m+1} + V_{m+1})$. Nevertheless, the following sharper bound holds:

THEOREM 4.1. *Under the assumptions stated in Lemma* 4.2 *the error* $\Delta w^{(m)}$ *in the asymptotic expansion* (29) *is bounded by*

$$\max_{0\le t\le T}\ \sup_{x\in\mathscr{R}} \left|\Delta w^{(m)}(\cdot,\cdot,\delta)\right| \le C\delta^{m+1}.$$

Again, the constant C depends on m, but not on δ..

Proof. Let $\Delta\bar v = \delta^{m+1}(v_{m+1} + V_{m+1})$. Clearly, by Lemma 4.2:

$$M_j(\Delta\bar v(\cdot,\cdot,\delta)) \le C\delta^{m+1}.$$

Now we study $\Delta\hat v$ given by $\Delta v^{(m)} = \Delta\hat v + \Delta\bar v$. The system for $\Delta u^{(m)}$ and $\Delta\hat v$ reads

$$\begin{aligned} \Delta u_t^{(m)} + A_{11}\Delta u_x^{(m)} + A_{12}\Delta\hat v_x &= \delta^{m+1}\,\partial_t U_{m+1} - A_{12}\Delta\bar v_x, \\ \Delta\hat v_t + A_{21}\Delta u_x^{(m)} + A_{22}\Delta\hat v_x &= \frac{1}{\delta}B_{22}\Delta\hat v - \delta^{m+1}\,\partial_t v_{m+1} - A_{22}\Delta\bar v_x, \\ \Delta u^{(m)}(x,0,\delta) = \Delta\hat v(x,0,\delta) &= 0. \end{aligned}$$

Lemma 4.1 applied to this system yields

$$M_j\left(\begin{pmatrix}\Delta u^{(m)} \\ \Delta\hat v\end{pmatrix}\right) \le C\left\{\delta^{m+1} I_j\left(\partial_t\begin{pmatrix}U_{m+1} \\ v_{m+1}\end{pmatrix}\right) + I_j(\Delta\bar v_x)\right\}.$$

Again, by Lemma 4.2, $I_j(\Delta\bar v_x)$ is of the desired order $I_j(\bar\partial_x) \le I_{j+1}(\Delta\bar v) \le C\delta^{m+1}$ and

$$I_j(\partial_t v_{m+1}) \le TM_j(\partial_t v_{m+1}) \le C.$$

Finally,

$$I_j(\partial_t U_{m+1}) = \int_0^T \left\| \partial_x^j \partial_t U_{m+1}\left(\cdot, \frac{t}{\delta}\right) \right\| dt \le \frac{C}{\delta} \int_0^T \sum_{l=0}^m \left(\frac{t}{\delta}\right)^l e^{-\gamma t/\delta}\, dt$$
$$\le C \int_0^{T/\delta} \sum_{l=0}^m \tau^l e^{-\gamma\tau}\, d\tau \le C.$$

Now it is obvious that

$$M_j\big(\Delta w^{(m)}(\cdot, \cdot, \delta)\big) \le C\delta^{m+1}.$$

Because this bound holds for any $j \in \mathcal{N}_0$, Sobolev's inequality yields maximum norm estimates. This proves the theorem.

The theorem and its proof directly generalize to any number of space dimensions. Also, the introduction of a δ-independent term Cu would cause no problems.

5. A MODEL FOR TWO-PHASE FLOW

In this section we study a linearized version of the polymer flooding system (2):

$$\begin{aligned} s_t + f(s, c)_x &= 0, \\ (cs + a)_t + (cf(s, c))_x &= 0, \\ \delta a_t &= E(c) - a, \qquad 0 < \delta \ll 1. \end{aligned}$$

To apply the theory of stiff well-posedness, we linearize the system at a constant state $(\bar{s}, \bar{c}, \bar{a})$ with $\bar{a} = E(\bar{c})$. We substitute the ansatz,

$$s = \bar{s} + \sigma, \qquad c = \bar{c} + \gamma, \qquad a = \bar{a} + \alpha,$$

into the system and neglect quadratic terms. The result is a linear constant coefficient system for the deviation $u = (\sigma, \gamma, \alpha)^T$. To simplify the notation, superscript bars shall be omitted. We write s for $\bar{s}$ and let $f = f(\bar{s}, \bar{c})$, $f_s = f_s(\bar{s}, \bar{c})$, $f_c = f_c(\bar{s}, \bar{c})$, and $e = E'(\bar{c})$. Then the linearized system reads

$$u_t + \begin{pmatrix} f_s & f_c & 0 \\ 0 & \dfrac{f}{s} & 0 \\ 0 & 0 & 0 \end{pmatrix} u_x = \frac{1}{\delta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\dfrac{e}{s} & s^{-1} \\ 0 & e & -1 \end{pmatrix} u. \tag{30}$$

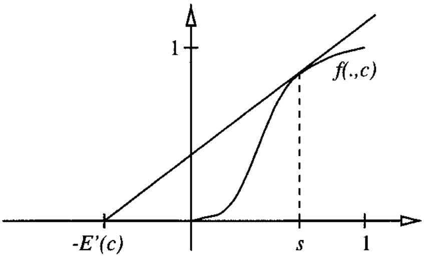

FIG. 1. A typical fractional flow function f.

Here the numbers f, s, f_s, and e are real and positive, and f_c is real. If $f_s \neq f/s$, this system is strongly hyperbolic. The spectrum of B is $\sigma(B) = \{0, 0, \lambda_3 = -1 - e/s\}$, and therefore condition C3, formulated in Section 2, is fulfilled. The system in canonical block form reads

$$v_t + Cv_x = \frac{1}{\delta}Dv, \tag{31}$$

where

$$C := \begin{pmatrix} f_s & f_c & -f_c \\ 0 & b & -b \\ 0 & -\dfrac{be}{s} & \dfrac{be}{s} \end{pmatrix}, \qquad b := \frac{f}{e+s}, \quad \text{and} \quad D := \operatorname{diag}(0, 0, \lambda_3).$$

The equilibrium system is

$$\breve{v}_t^I + \begin{pmatrix} f_s & f_c \\ 0 & b \end{pmatrix} \breve{v}_x^I = 0. \tag{32}$$

If $f_c \neq 0$, then critical situations (regarding well-posedness) occur when either $f_s = f/s$ or $f_s = b$, because the system (30) or (32) is not strongly hyperbolic, respectively. Due to the S-shape of $f(\cdot, c)$, for any given c there is a critical value for s; see Fig. 1. The next lemma gives precise conditions for the stiff well-posedness of System (30).

LEMMA 5.1. *Consider the stiff hyperbolic system* (30) *where all coefficients are real and*

$$s > 0, \qquad f > 0, \qquad f_s > 0, \qquad e > 0, \qquad 0 < \delta \ll 1.$$

The well-posedness and stiff well-posedness of (30) *is determined by the parameters as follows.*

1. *If* $f_c = 0$, *the Cauchy problem is stiffly well-posed.*

2. *If* $f/s \neq f_s \neq b := f/(e+s)$, *the Cauchy problem is stiffly well-posed.*

3. *If* $f_c \neq 0$, $f_s \neq f/s$ *but* $f_s = b$, *the Cauchy problem is well-posed for each* δ, *but not stiffly well-posed.*

4. *If* $f_c \neq 0$ *and* $f_c = f/s$, *the system is not strongly hyperbolic. Therefore, the Cauchy problem is not well-posed and, in particular, not stiffly well-posed.*

Except for case 2, which is the generic case, these statements follow directly from earlier results. To see this, we first discuss the exceptional cases 1, 3, and 4.

Case 1. Consider the system in canonical block form (31). By assumption we have $f_c = 0$. The matrix $H = \operatorname{diag}(1, e/s, 1)$ is a symmetrizer for

$$C = \begin{pmatrix} f_s & 0 & 0 \\ 0 & b & -b \\ 0 & -\dfrac{be}{s} & \dfrac{be}{s} \end{pmatrix};$$

i.e., HC is symmetric. Because $D = HD$ and $D + D^* = 2D \leq 0$, the simple criterion Theorem 2.1 yields stiff well-posedness.

In *Case 3* the full system (30) is strongly hyperbolic, but the equilibrium system (32) is not strongly hyperbolic. By Theorem 2.3, C1 does not hold; i.e., (30) is not stiffly well-posed.

Under the conditions of *Case 4* System (30) is not strongly hyperbolic. Therefore, the Cauchy problem is not well-posed.

It is more involved to show stiff well-posedness under the assumptions $f/s \neq f_s \neq b$ in *Case 2*. We prove the following slightly more general result.

LEMMA 5.2. *The Cauchy problem for the system*

$$u_t + \begin{pmatrix} \alpha & * & * \\ 0 & \beta & -\beta \\ 0 & -\gamma & \gamma \end{pmatrix} u_x = \frac{1}{\delta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_{33} \end{pmatrix} u$$

with real coefficients α, β, γ, *and* b_{33} *is stiffly well-posed if the following conditions hold.*

(i) $b_{33} < 0$;

(ii) $0 \neq \alpha \neq \beta + \gamma \neq 0$;

(iii) $\alpha \neq \beta$;

(iv) $\beta\gamma > 0$.

Here the entries $*$ *are arbitrary real numbers.*

Proof. To prove the lemma, we apply Theorem 2.5. Obviously, $b_{33} < 0$ yields (a) in Theorem 2.5. By rescaling we may assume, without loss of generality, that $b_{33} = -1$. The eigenvalues of A are $\alpha, \beta + \gamma$ and zero. Hence condition (ii) states that the stiff system is strictly hyperbolic. Furthermore, the eigenvalues of the reduced system are α and β, and therefore (iii) is equivalent to strict hyperbolicity of the reduced system. The main point is to show that the assumption $\beta\gamma > 0$ implies condition (d) in Theorem 2.5. This will be shown next.

The eigenvalues of $P = B - i\omega A$ are $-i\omega\alpha$, q_+, and q_-, where $q_\pm$ are the eigenvalues of

$$Q := \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} - i\omega \begin{pmatrix} \beta & -\beta \\ -\gamma & \gamma \end{pmatrix}.$$

Observe that it is sufficient to show $\operatorname{Re} q_\pm < 0$ for $\omega \neq 0$. The characteristic equation for Q is

$$q^2 + q(1 + i\omega(\beta + \gamma)) + i\omega\beta = 0,$$

and the eigenvalues are

$$q_\pm = -\tfrac{1}{2}(1 + i\omega(\beta + \gamma)) \pm \tfrac{1}{2}\Big[(1 + i\omega(\beta + \gamma))^2 - 4i\omega\beta\Big]^{1/2}.$$

Let us consider small $|\omega| \neq 0$ first. Clearly, we have $q_- = -1 + \mathscr{O}(|\omega|)$, and therefore

$$\operatorname{Re} q_- < 0,$$

if $|\omega|$ is small enough. The second eigenvalue is

$$q_+ = -\tfrac{1}{2}(1 + i\omega(\beta + \gamma)) + \tfrac{1}{2}\Big[(1 + 2i\omega(\gamma - \beta)) - \omega^2(\beta + \gamma)^2\Big]^{1/2}.$$

Using $\sqrt{1+\epsilon} = 1 + \frac{\epsilon}{2} - \epsilon^2/8 + \mathscr{O}(\epsilon^3)$ as $\epsilon \to 0$, we expand the root and find that

$$\begin{aligned}\operatorname{Re} q_+ &= \frac{1}{2} + \frac{1}{2}\left[1 - \frac{\omega^2}{2}(\beta+\gamma)^2 + \frac{\omega^2}{2}(\gamma-\beta)^2\right] + \mathscr{O}(|\omega|^3)\\ &= -\omega^2\beta\gamma + \mathscr{O}(|\omega|^3), \quad |\omega| \to 0.\end{aligned}$$

Therefore, the assumption $\beta\gamma > 0$ implies that

$$\operatorname{Re} q_+ < 0$$

for small $|\omega| \neq 0$.

Next we show that $\operatorname{Re} q_\pm \neq 0$ for $|\omega| \neq 0$. To this end, let $q_\pm = x_\pm + iy_\pm$ with $x_\pm, y_\pm \in \mathscr{R}$. From $\det Q = i\omega\beta = q_+ q_-$ it follows that

$$x_+ x_- = y_+ y_-, \tag{33}$$

$$x_+ y_- + x_- y_+ = \omega\beta. \tag{34}$$

From $\operatorname{trace} Q = -1 - i\omega(\beta+\gamma) = q_+ + q_-$ we obtain that

$$x_+ + x_- = -1, \tag{35}$$

$$y_+ + y_- = -\omega(\beta+\gamma). \tag{36}$$

From (iv) we have $\beta \neq 0$, hence $\det Q \neq 0$ if $\omega \neq 0$. Therefore, $q_+ \neq 0 \neq q_-$. Now suppose that $\operatorname{Re} q_+ = x_+ = 0$, say. Then $y_+ \neq 0$, and (33) implies $y_- = 0$. Furthermore, (35) yields $x_- = -1$, and by (34) one obtains $y_+ + y_- = -\omega\beta$. But this contradicts (36) and $\gamma \neq 0$, as follows from (iv). Therefore, $x_+ = \operatorname{Re} q_+ \neq 0$, and the same argument applies to $x_- = \operatorname{Re} q_-$. To summarize, we have shown that

$$\operatorname{Re} q_\pm < 0 \quad \text{for all } \omega \neq 0,$$

and the lemma is proved.

We now show that no constant symmetrizer exists for (30) if $f_c \neq 0$. Let us start with a simple observation: If there is a constant symmetrizer H_0 for a system $u_t + Au_x = \frac{1}{\delta}Bu$ and one transforms variables $u = Tv$ to obtain

$$v_t + A'v_x = \frac{1}{\delta}B'u, \qquad A' = T^{-1}AT, \qquad B' = T^{-1}BT,$$

then $H_0' = T^*H_0T$ is a constant symmetrizer for the transformed system. Therefore, instead of (30) we may consider (31). Then, after scaling, the following result applies.

LEMMA 5.3. *Consider a system $u_t + Au_x = \frac{1}{\delta}Bu$ with*

$$A = \begin{pmatrix} \alpha & \sigma & -\sigma \\ 0 & \beta & -\beta \\ 0 & -\gamma & \gamma \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with real coefficients and $\alpha \neq 0$, $\sigma \neq 0$. A matrix $H = H^* > 0$ with

$$HA = A^TH \quad \textit{and} \quad HB + BH \leq 0$$

does not exist.

Proof. Assume $H = H^*$ has the properties stated. It is not difficult to show that $H = H^*$ and $HB + BH \leq 0$ imply that H has block structure,

$$H = \begin{pmatrix} x & z & 0 \\ \bar{z} & y & 0 \\ 0 & 0 & h_{33} \end{pmatrix}.$$

(Use, for example, Lemma 2.1 of [10].) Without loss of generality, $h_{33} = 1$. Then one obtains

$$HA = \begin{pmatrix} \alpha x & \sigma x + \beta z & -\sigma x - \beta z \\ \alpha \bar{z} & \sigma \bar{z} + \beta y & -\sigma \bar{z} - \beta y \\ 0 & -\gamma & \gamma \end{pmatrix}.$$

From $HA = (HA)^*$ one obtains that

$$\sigma x + \beta z = \alpha \bar{z}, \qquad -\sigma x - \beta z = 0.$$

Because $\alpha \neq 0$, this yields $z = 0$. Therefore, $\sigma x + \beta z = \alpha \bar{z} = 0$, and $\sigma \neq 0$ implies $x = 0$. This contradicts $H > 0$, and the lemma is proved.

To complete the paper, we present results of a numerical experiment illustrating stiff well-posedness and the asymptotic expansion for the linearized model (30). It will be assumed that

$$\frac{f}{s} \neq f_s \neq \frac{f}{e + s};$$

i.e., we are in the generic case, case 2, of Lemma 5.1. For the experiment we have chosen the parameters $f = \frac{1}{3}$, $s = \frac{1}{3}$, $f_s = \frac{1}{3}$, $f_c = 1$, and $e = \frac{1}{3}$.

Then, by Lemma 5.1, the system is stiffly well-posed. We have computed the solution of the Cauchy problem for the blocked system (31) as well as the leading- and first-order approximations. The initial data is $v(x, 0, \delta) = (\sin(\pi x), \cos(\pi x), 0)^T$. Because the data are in the null-space of the relaxation term D, we expect only a small initial layer (with initial amplitude of order δ). The computations are performed in the space interval $[0, 2]$. At $x = 0$ and $x = 2$ periodic boundary conditions where imposed. The PDEs were discretized by first-order upwind differences for the convective terms and a trapezoidal rule for the time integration of the forcing. The space derivative in the forcing terms was approximated by central differences. The mesh sizes in space and time were uniform, $\Delta x = \Delta t = \frac{1}{800}$.

Figure 2 shows the first component of the solution, while Figs. 3 and 4 show the leading- and first-order errors, respectively. Here, the parameter δ was fixed at $\delta = \frac{1}{10}$.

Figure 5 clearly shows first- and second-order convergence of the leading- and first-order approximations as δ tends to zero and thus confirms the theory.

6. CONCLUDING REMARKS

The emphasis in the papers [1, 9, 10] is on nonlinear problems, whereas we have considered only linear constant coefficient problems in this paper. For this restricted class of problems our results are more complete, however. In particular, Theorem 2.2 gives conditions for convergence that

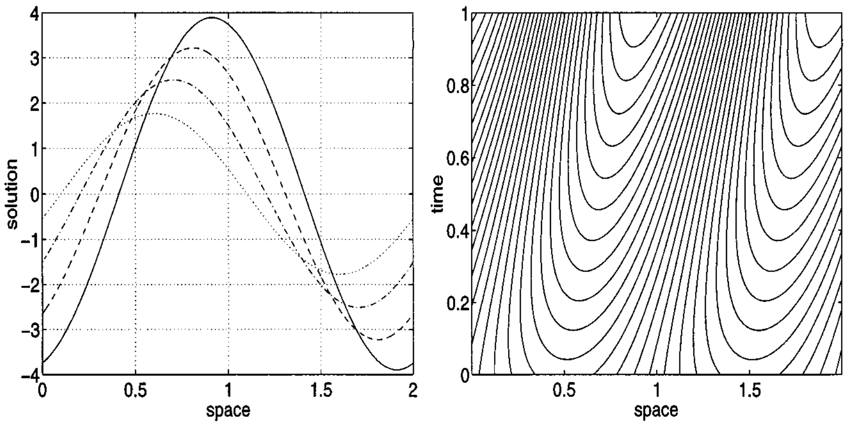

FIG. 2. First component of the solution of (31). Left side: (Dotted line) $t = \frac{1}{4}$, (dash-dotted) $t = \frac{1}{2}$, (dashed) $t = \frac{3}{4}$, (solid) $t = 1$.

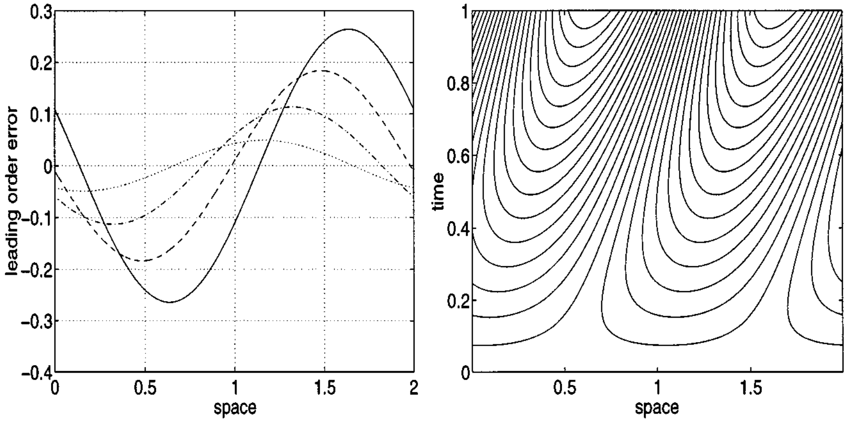

FIG. 3. Error of the leading-order approximation ($\Delta v_1^{(0)} = v_1 - u_1^{(0)}$). Left side: (Dotted line) $t = \frac{1}{4}$, (dash-dotted) $t = \frac{1}{2}$, (dashed) $t = \frac{3}{4}$, (solid) $t = 1$.

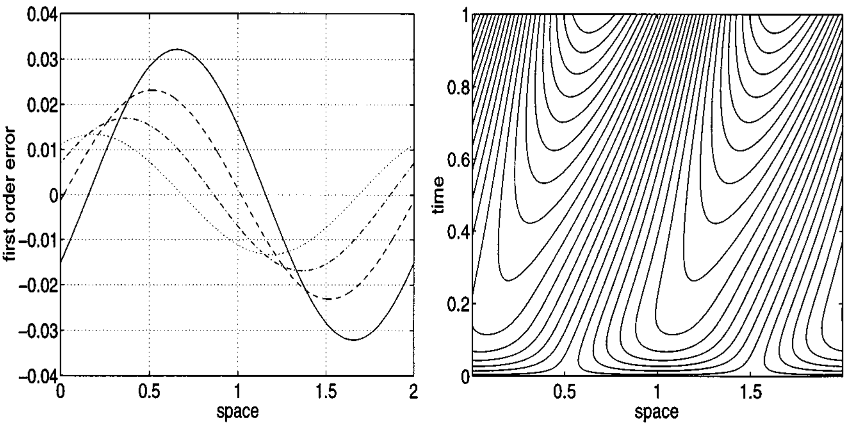

FIG. 4. Error of the first-order approximation ($\Delta v_1^{(1)} = v_1 - v_1^{(0)} - \delta v_1^{(1)}$). Left side: (Dotted line) $t = \frac{1}{4}$, (dash-dotted) $t = \frac{1}{2}$, (dashed) $t = \frac{3}{4}$, (solid) $t = 1$. Right side: A small initial layer can be observed.

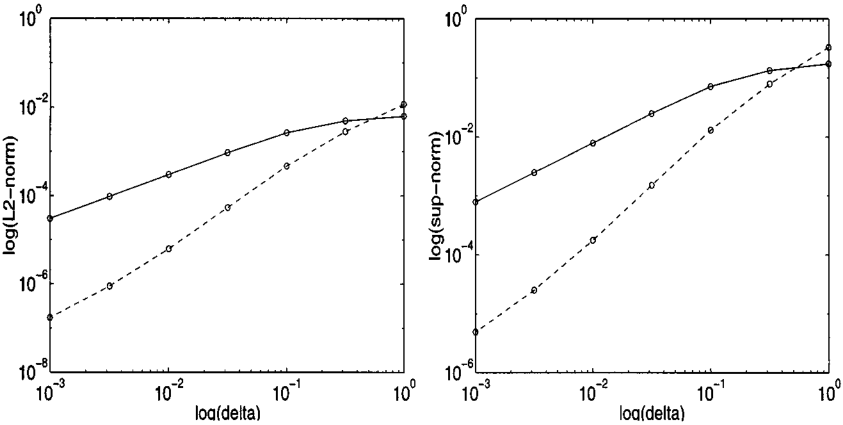

FIG. 5. Convergence rates. (Solid line) $\|\Delta v^{(0)}(t, \cdot, \delta)\| = \mathcal{O}(\delta)$, (dashed line) $\|\Delta v^{(1)}(t, \cdot, \delta)\| = \mathcal{O}(\delta^2)$, both at time $t = \frac{1}{8}$.

are necessary and sufficient. This leads to the concept of stiff well-posedness. Previous results have only dealt with sufficient conditions for convergence.

As pointed out in [1, 9, 10], the existence of an entropy, which is a basic assumption in [1, 10], is fulfilled in many physical examples. A main point of our paper is to present a physical example which does not have an entropy, but is stiffly well-posed, nevertheless. Stiff well-posedness is sufficient to show desired properties, such as the validity of an asymptotic expansion. (The formal process of obtaining the expansion is standard.)

In future work we will extend our approach, based on L_2 estimates, Fourier transformation, and symmetrizers $H(\omega)$, to linear problems with variable coefficients and also to nonlinear problems. Such an extension is possible as long as the solution stays smooth. In the case of linear variable coefficient problems, the idea is to construct a symmetrizer $H(x, t, \omega)$ for all frozen coefficient problems and to apply the techniques of pseudo-differential operators with symbol H. For nonlinear problems, the symmetrizer will depend on the solution itself.

Of course, the treatment of nonlinear problems with shocks is of major interest. At present it is not clear if and how our approach (based on L_2) must be modified.

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